Mechanical Models with Interval Parameters*

E. D. Popova[1], M. Datcheva[2,3], R. Iankov[2], and T. Schanz[3]

[1] Inst. of Mathematics & Informatics, Bulgarian Academy of Sciences
[2] Institute of Mechanics, Bulgarian Academy of Sciences
Acad. G. Bonchev str., block 8[1], block 4[2], BG-1113 Sofia, Bulgaria
epopova@bio.bas.bg, {iankovr, datchevam}@yahoo.com
[3] Bauhaus-Universität Weimar, Germany, tom.schanz@bauing.uni-weimar.de

Abstract. In this paper we consider modelling of composite material with inclusions where the elastic material properties of both matrix and inclusions are uncertain and vary within prescribed bounds. Such mechanical systems, involving interval uncertainties and modelled by finite element method, can be described by parameter dependent systems of linear interval equations and process variables depending on the system solution. A newly developed hybrid interval approach for solving parametric interval linear systems is applied to the considered model and the results are compared to other interval methods. The hybrid approach provides very sharp bounds for the process variables – element strains and stresses. The sources for overestimation when dealing with interval computations are demonstrated. Based on the element strains and stresses, we introduce a definition for the values of nodal strains and stresses by using a set-theoretic approach.

1 Introduction

All engineering design problems involve imprecision, approximation, or uncertainty to varying degrees [2, 4]. In particular, mathematical models in environmental geomechanics cover a broad class of problems involving uncertainties of different types. Since soil and rock materials are natural ones, there is uncertainty in the material properties [7]. When the information about an uncertain parameter in form of a preference or probability function is not available or not sufficient then the interval analysis can be used most conveniently [4]. Many mechanical systems, modelled by finite element method (FEM), can be described by parameter dependent systems of linear equations. If some of the parameters are uncertain but bounded, the problem can be transformed into a parametric interval linear system which should be solved appropriately to bound the mechanical system response. This technique is usually called Interval Finite Element Method. The efforts for developing suitable interval FE methods started at mid nineties and attract considerable attention [2]. Here, we consider a a 2D plane strain problem with two inclusions and interval parameters related to the elastic material properties of both matrix and inclusions. Since safety is an issue in environmental geomechanics, the goal is to describe the response of the system under a worst case scenario of uncertain parameters varying within prescribed bounds.

2 FEM model with interval uncertainties

Let us consider an elastic material model based on the following assumptions: small strain theory is applied to describe the deformation in material; the latter is deformed elastic; material properties are isotropic; the temperature, creep and time dependent effects are not taken into account. Let V be a representative volume and S be the boundary of this volume. $S = S_{\bar{u}} \bigcup S_\sigma$, where $S_{\bar{u}}$ is a boundary part with prescribed displacement, and S_σ is a boundary part with prescribed stresses. The governing equations in a 3D case are as follows:

$$\begin{aligned} &\text{Kinematics :} && \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) && \text{in } V \\ &\text{Equilibrium :} && \sigma_{ij,j} + b_i = 0 && \text{in } V \end{aligned} \qquad (1)$$

* This work was supported by the NATO CLG 979541 and the Bulgarian National Science Fund under grant No. I-903/99.

$$\text{Constitutive Law :} \qquad d\sigma_{ij} = D_{ijkl}\, d\varepsilon_{kl} \qquad \text{in } V$$

$$\text{Boundary Cond. :} \qquad \mathrm{u_i = \bar{u}_i \quad on \quad S_{\bar{u}}}, \qquad \mathrm{\sigma_{ij} n_j = \bar{T}_i \quad on \quad S_\sigma} \ ,$$

where u_i, $i = 1, 2, 3$, are the displacements, ε_{ij} is a strain tensor, b_i is the body force, σ_{ij} is the stress tensor, D_{ijkl} is the elastic tensor, $\bar{u}_i$ and $\bar{T}_i$ are the prescribed displacements and distributed load respectively, $n = \{n_1, n_2, n_3\}$ is the outward normal direction and n_i are the corresponding directional cosines.

The equilibrium equation (1), relating the stress vector $\{\sigma\}$ to the body force vector $\{b\}$ and the boundary force specified at the boundary S_σ of V, is formulated in terms of the unknown displacement vector $\{u\}$. Using the principle of virtual work, the general equilibrium statement can be written in a variational form [8] as:

$$\int_V \{\delta\varepsilon\}^T \{\sigma\}\, dV - \int_V \{\delta u\}^T \{b\}\, dV - \int_{S_\sigma} \{\delta u\}^T \{\bar{T}\}\, dS = 0 \tag{2}$$

for a virtual displacement vector $\{\delta u\}$.

The resulting system of equations is created using mathematical formulations for FEM based on variational technique. A typical FE in the 2D plane strain case is defined by nodes i, j, k. Let the displacement at any point within the element $\{u\}$, be approximated by linear shape functions and nodal unknown displacement $\{u_e\}$. With displacements, known at all points within an element, the strains $\{\varepsilon\}$ and stresses $\{\sigma\}$ at any point of the FE can be determined by nodal unknowns. These relationships can be written in matrix notations [10] as:

$$\{u\} = [N] \cdot \{u_e\}, \qquad \{\varepsilon\} = [L] \cdot [N] \cdot \{u_e\} = [B] \cdot \{u_e\} \ , \tag{3}$$

$$\{\sigma\} = [D] \cdot \{\varepsilon\} = [D] \cdot [B] \cdot \{u_e\} \ , \tag{4}$$

where $[N]$ is a matrix of shape functions, $[L]$ is a matrix of a suitable differential operators, $[D]$ is the elastic matrix involving Elastic modulus and Poisson's ratio.

Substituting equations (3) and (4) into the equation (2) and taking into account that the equation (2) is valid for any virtual displacement, for an element e, we obtain a system of linear equations

$$[K^e] \cdot \{u_e\} = \{F^e\} \ , \tag{5}$$

where $[K^e] = \int_V [B]^\top [D][B]\, dV$ is the element stiffness matrix which is a function of the uncertain material properties and $\{F^e\} = \int_V [N]^\top \{b\}\, dV + \int_S [N]^\top \{\bar{T}\}\, dS$ is the element load vector. The final global stiffness matrix and the global loading vector based on FEM matrices (5) are obtained by using an element-by-element technique [10].

We consider models in which the Elastic modulus is uncertain and is varying within prescribed bounds. This case is of interest because the prediction of soil behavior is quite sensitive to measurement errors and in situ variations of the Elastic modulus. In the general case, within an element e, denote by p_e a parameter (or parameter vector), related to the material properties, that are considered as uncertain. Then the elastic matrix in (4), and the corresponding element stiffness matrix in (5) become parametric matrices $[D] = D(p_e)$, $[K^e] = K^e(p_e)$. The loading vector $\{F^e\}$ could also depend on some uncertain parameters. In order to avoid a subsequent confusion with the notations for the interval quantities, from now on we shall omit the square brackets for the matrices. Square brackets will be used only to denote interval quantities. For some of the vectorial variables and quantities the curly braces will be also omitted.

Like in FE analysis of most engineering systems [2, 4], the problem considered here can be stated as follows: Solve a system of linear equations, that describe the response of a discretized structural mechanical model, given in the form

$$K(p) \cdot U = F(p) \tag{6}$$

and define a response vector

$$R = R(U) \ , \tag{7}$$

where $K(p)$ is the global stiffness matrix of the system, $F(p)$ is the global loading vector (which might depend on uncertain parameters), U is the nodal displacement vector of the total discrete model, and p is the vector of uncertain input parameters which are described as intervals $p = \{p_i\} = \{[p_i^-, p_i^+]\}$, $i = 1, 2, \ldots, m$, where p_i^- denotes the lower bound and p_i^+ denotes the upper bound for the corresponding parameter value. R is the vector of response quantities which can be expressed in terms of nodal displacements, U. In this work we consider as response quantities the process variables, namely the element strain vector, ε_e (3), and the element stress vector, σ_e (4),

$$\varepsilon_e = B \cdot U_\mathrm{e} \qquad \text{and} \tag{8}$$

$$\sigma_\mathrm{e} = D(p_\mathrm{e}) \cdot B \cdot U_\mathrm{e} \ , \tag{9}$$

where p_e is the element parameter vector which is, generally, a subvector of the global parameter vector p, and U_e is the subvector of element nodal displacements. Because U_e depends on p, ε_e depends implicitly on p and σ_e depends both explicitly and implicitly on p. Thus there will be a range for each component of R, so that $R = [R^-,\, R^+] = \{[r_i^-, r_i^+]\}$. The problem is to find the range $[R^-,\, R^+]$ of each response quantity due to the uncertainty present in the input parameters. A key issue is how to avoid the overestimation effect caused by the dependency problem in interval analysis.

The solution of parametric linear interval equations (6) and subsequent sharp estimation of the response parameters (8), (9) is discussed in this work. We extend the hybrid approach for solving parametric interval linear systems [3] to the subsequent bounding of the process variables – the element strains and the element stresses.

3 Interval Methods

3.1 Solving Parametric Interval Linear Systems

Linear algebraic systems (6) usually involve complicated dependencies between the parameters involved in the stiffness matrix $K(p)$ and the load vector $F(p)$. In this paper we assume that these dependencies are affine-linear, that is

$$k_{ij}(p) = \lambda_{ij0} + \sum_{\nu=1}^{m} \lambda_{ij\nu} p_\nu, \qquad f_i(p) = \beta_{i0} + \sum_{\nu=1}^{m} \beta_{i\nu} p_\nu \ ,$$

where $\lambda_{ij}, \beta_i \in \mathbb{R}^{m+1}$ $(i, j = 1, \ldots, n)$ are numerical vectors, with n being the dimension of (6). When the m parameters $p_\nu, (\nu = 1, \ldots, m)$ take arbitrary values from given intervals $[p_\nu]$, the solution of (6) is a set

$$\Sigma^p = \Sigma\left(K(p), F(p), [p]\right) := \{U \in \mathbb{R}^n \mid K(p) \cdot U = F(p) \text{ for some p} \in \text{[p]}\}$$

called parametric solution set (PSS). The PSS is a subset of, and has much smaller volume than the corresponding non-parametric solution set

$$\Sigma^g = \Sigma([K], [F]) := \{U \in \mathbb{R}^n \mid \exists K \in [K] = K([p]), \exists F \in [F] = F([p]) : K \cdot U = F\}.$$

The simplest example of dependencies is when the matrix is symmetric. Since the solution sets have a complicated structure which is difficult to find, we look for the interval hull $\square\Sigma := [\inf \Sigma,\ \sup \Sigma]$, whenever Σ is a nonempty bounded subset of $\mathbb{R}^n$, or for an interval enclosure $[U]$ of $\square\Sigma$.

There are many works devoted to interval treatment of uncertain mechanical systems [2]. A main reason for some conservative results that have been obtained is that the interval linear systems (6), involving more dependencies than in a symmetric matrix, are solved by methods for nonparametric interval systems (or for symmetric matrices). Although designed quite long ago, the only available iterative method [6] for guaranteed enclosure of the PSS seems to be not known to the application scientists, or at least not applied. The parametric Rump's method is a fixed-point method applying residual iteration and accounting for all the dependencies between parameters. As it will be shown in Sec. 4, this *parametric* method produces very sharp bounds for narrow intervals but tends to overestimate the PSS hull with increasing the interval tolerances and the number of parameters. To overcome this deficiency, we designed a new hybrid approach for sharp PSS enclosures, that combines parametric residual iteration and exact bounds based on monotonicity properties [3].

Finding very sharp bounds (or exact in exact arithmetic) for the PSS is based on the monotonicity properties of the analytic solution $U(p) = K(p)^{-1} \cdot F(p)$. For large real-life problems, the computer aided proof of the monotonicity of $U(p)$ with respect to each parameter is based on taking partial derivatives on (6) [5]. This leads to a parametric interval linear system

$$K(p) \frac{\partial U}{\partial p_\nu} \ = \ \frac{\partial F(p)}{\partial p_\nu} \ - \ \frac{\partial K(p)}{\partial p_\nu} \cdot [\tilde{U}] \ , \tag{10}$$

where $[\tilde{U}] \supseteq \Sigma^p$ is a PSS enclosure. Let for fixed i, $1 \leq i \leq n$

$$L_-^{u_i} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \mathrm{U_i}}{\partial \mathrm{p}_\nu}\right]) = 1\}, \qquad L_+^{u_i} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \mathrm{U_i}}{\partial \mathrm{p}_\nu}\right]) = -1\}$$

and $L_-^{u_i} \cup L_+^{u_i} = \{1, \ldots, m\}$. Define numerical vectors p^{u_i}, p^{-u_i} componentwise

$$p_j^{u_i} := \begin{cases} p_j^- & \text{if } \mathrm{j} \in \mathrm{L_-^{u_i}}, \\ p_j^+ & \text{if } \mathrm{j} \in \mathrm{L_+^{u_i}} \end{cases} \quad \text{and} \quad \mathrm{p_j^{-u_i}} := \begin{cases} p_j^+ & \text{if } \mathrm{j} \in \mathrm{L_-^{u_i}}, \\ p_j^- & \text{if } \mathrm{j} \in \mathrm{L_+^{u_i}} \end{cases} \quad \mathrm{j} = 1, \ldots, \mathrm{m} \ . \tag{11}$$

Then the exact bounds of the PSS, $U_i^- = \inf\{\Sigma^p\}_i$ and $U_i^+ = \sup\{\Sigma^p\}_i$,

$$[U_i^-, U_i^+] \ = \ [K(p^{u_i})^{-1} \cdot F(p^{u_i}),\ K(p^{-u_i})^{-1} \cdot F(p^{-u_i})], \qquad i = 1, \ldots, n$$

can be obtained by solving at most $2n$ point linear systems. This approach is extended in the next subsection for sharp bounding of the process variables depending on the parametric system solution.

3.2 Bounding the Process Variables

A naive interval approach in finding the range of the process variables $R(U)$ under the uncertainties in input parameters is to replace the exact hull (or its enclosure) $[U]$, found by some method, into the expressions (8), (9) and to perform all interval operations. Since the nonzero components of $[U_e]$ depend on the uncertain parameters, the range computation of $R(U_e)$ implicitly involves the dependency problem implying a huge overestimation of the exact range, as will be shown in Sec. 4. To get sharp range estimations, we use the monotonicity properties of the responses. Taking partial derivatives on (8), (9) we obtain

$$\frac{\partial \varepsilon_e}{\partial p_\nu} = B \cdot \left[\frac{\partial U_e}{\partial p_\nu}\right] \qquad (\text{since } B \text{ doesn't depend on } p) \tag{12}$$

$$\frac{\partial \sigma_e}{\partial p_\nu} = \frac{\partial D(p_e)}{\partial p_\nu} \cdot [\varepsilon_e(p)] + D(p_e) \cdot \left[\frac{\partial \varepsilon_e}{\partial p_\nu}\right] \qquad \nu = 1, \ldots, m \ , \tag{13}$$

where $\left[\frac{\partial U_e}{\partial p_\nu}\right]$ is a subvector of the solution of (10), $[\varepsilon_e(p)]$ is the solution of (8), and $\left[\frac{\partial \varepsilon_e}{\partial p_\nu}\right]$ is taken from (12). Let for fixed $i \in \{1, \ldots, n_1\}$, $j \in \{1, \ldots, n_2\}$, where n_1, n_2 are the dimensions of ε_e, σ_e respectively.

$$L_-^{\varepsilon_i} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \{\varepsilon_\mathrm{e}\}_\mathrm{i}}{\partial \mathrm{p}_\nu}\right]) = 1\}, \qquad L_+^{\varepsilon_i} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \{\varepsilon_\mathrm{e}\}_\mathrm{i}}{\partial \mathrm{p}_\nu}\right]) = -1\}$$

$$L_-^{\sigma_j} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \{\sigma_\mathrm{e}\}_\mathrm{j}}{\partial \mathrm{p}_\nu}\right]) = 1\}, \qquad L_+^{\sigma_j} = \{\nu \mid \mathrm{Sign}(\left[\frac{\partial \{\sigma_\mathrm{e}\}_\mathrm{j}}{\partial \mathrm{p}_\nu}\right]) = -1\} \ .$$

Note, that the sets L_-^u, L_+^u, $L_-^\varepsilon, L_+^\varepsilon, L_-^\sigma, L_+^\sigma$ differ between themselves and for every fixed i, j. Define the numerical vectors p^{ε_i}, $p^{-\varepsilon_i}$, for every $1 \leqq i \leqq n_1$, and p^{σ_j}, $p^{-\sigma_j}$, for every $1 \leqq j \leqq n_2$, analogously to (11). Then the exact bounds of ε_e are obtained as

$$[\{\varepsilon_e\}_i^-,\ \{\varepsilon_e\}_i^+] \ = \ [\{B\}_i \cdot U_e(p^{\varepsilon_i}),\ \{B\}_i \cdot U_e(p^{-\varepsilon_i})], \qquad i = 1, \ldots, n_1 \ .$$

Note, that a rigorous very sharp enclosure can be obtained if instead of $U_e(p^{\varepsilon_i})$, $U_e(p^{-\varepsilon_i})$ we take the corresponding components of the rigorous interval enclosures $[U(p^{\varepsilon_i})]$, $[U(p^{-\varepsilon_i})]$ of the solution to the corresponding point systems (6).

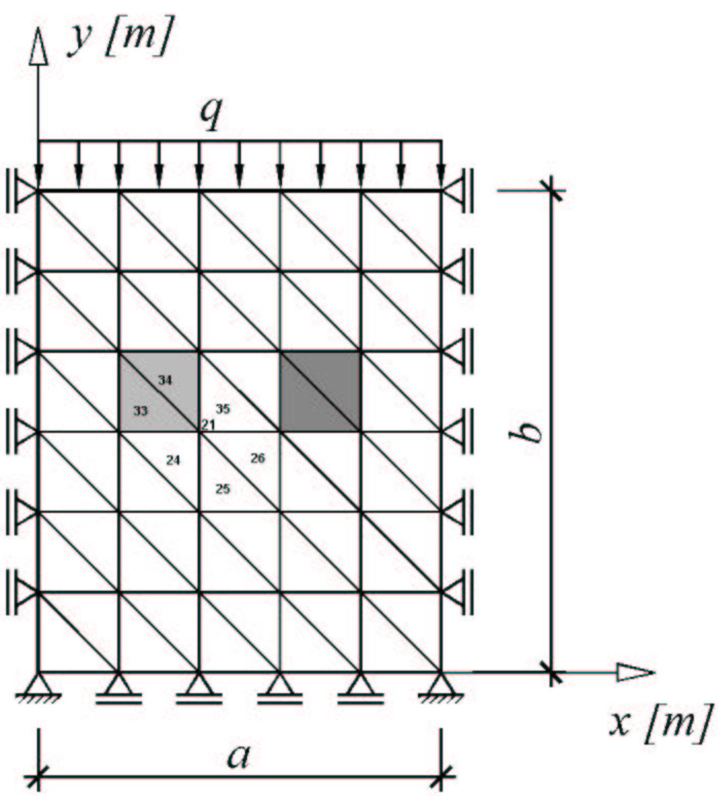
For fixed $j \in \{1, \ldots, n_2\}$

$$\inf\{\sigma_e\}_j = \{D(p_e^{\sigma_j}) \cdot B \cdot U_e(p^{\sigma_j})\}_j \ = \ \{D(p_e^{\sigma_j}) \cdot \varepsilon(p_e^{\sigma_j})\}_j$$
$$\sup\{\sigma_e\}_j = \{D(p_e^{-\sigma_j}) \cdot B \cdot U_e(p^{-\sigma_j})\}_j \ = \ \{D(p_e^{-\sigma_j}) \cdot \varepsilon(p_e^{-\sigma_j})\}_j \ .$$

However, the above computation of the σ_e range is rigorous only in exact arithmetic. A rigorous and very sharp range enclosure of σ_e in floating-point arithmetic should use rigorous interval enclosures $[U(p^{\sigma_j})]$, $[U(p^{-\sigma_j})]$ of the solution to the corresponding point linear systems (6) and computations with validated interval arithmetic.

4 Numerical Results

As a numerical example we consider a 2D plane strain problem, presented on Fig. 1, with two square inclusions. The geometry and boundary conditions are given on the figure. The area is modelled with 60 linear triangular finite elements. Each inclusion is approximated with two finite elements and the material properties of the inclusions are different and differ from the material properties of the base material, see Fig. 1. We consider the case when the Elastic moduli of the base material and of the inclusions are uncertain parameters E_i, $i = 1, 2, 3$. Therefore, the vector of parameters p is defined as $p = ([E_1 \pm \Delta\%], [E_2 \pm \Delta\%], [E_3 \pm \Delta\%])^\top$, where Δ is a prescribed tolerance value. The MATLAB Symbolic Toolbox [1] is used to generate the final parametric set of equations and to incorporate the boundary conditions. The aim of this example is to illustrate the proposed interval techniques for solving parametric interval linear systems (6) and a subsequent bounding the range of the response parameters (8), (9).



Young's modulus
$E_{\text{base}} = 200$ GPa $E_{\text{incl}_1} = 210$ GPa $E_{\text{incl}_2} = 220$ GPa
Poisson's ratio
$\nu_{\text{base}} = 0.3$ $\nu_{\text{incl}_1} = 0.29$ $\nu_{\text{incl}_2} = 0.28$

Fig. 1. 2D plane strain FEM model with 60 finite elements, where $a = 50m$, $b = 60m$, $q = 1GPa$, with two inclusions having different material properties

4.1 Comparison of Parametric Solvers

We solve the parametric interval linear system (6), where the vector of parameters p is defined as $p = ([E_1 \pm \Delta\%], [E_2 \pm \Delta\%], [E_3 \pm \Delta\%])^\top$, and Δ is the degree of uncertainty measured in % from the corresponding nominal value, given on Fig. 1. Different values for the tolerance Δ are considered, $\Delta = 0.01\%, 0.1\%, 1\%, 2\%, \ldots, 10\%$, in order to demonstrate how the range of the displacements vary with the varying degree of uncertainty. The results are graphically presented on Fig. 2, where the solid lines represent the corresponding lower, resp. upper bound for the parametric solution set, that is the exact range of the displacements. The degrees of freedom 42 and 29 are presented, DOF 42 corresponds to the node 21 (see Fig. 1) which is considered in more details, and DOF 29 is affected most by the overestimation.

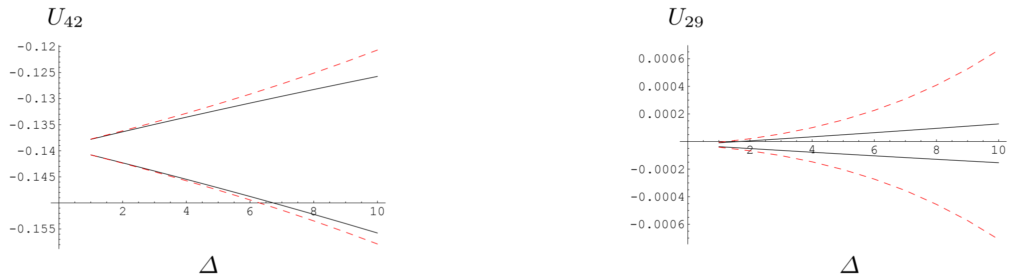


Fig. 2. For the 2D plane strain FEM model with varying degree of uncertainties in the material properties, the exact hull of the displacement solution set (solid line) and the enclosure obtained by the Rump's iteration method (dashed line).

A *Mathematica* [9] package `IntervalComputations‘LinearSystems‘` [3], providing a variety of functions for solving parametric and nonparametric interval linear systems in validated interval arithmetic, and supporting the hybrid interval approach, is used for the computations. We compare the results, obtained by the parametric iteration method, to the exact interval hull of the displacements. The overestimation of Rump's enclosure is measured in % as $100\,(1 - \omega(\square\Sigma^p)/\omega([U]))$, where $\square\Sigma^p$ is the exact hull of the corresponding parametric solution set, $[U]$ is the interval enclosure, obtained by the parametric iteration method, and $\omega(\cdot)$ is the width of an interval, defined as $\omega([a_-,\, a^+]) = a^+ - a^-$. The percentage by which the iterative solution enclosure overestimates the exact hull of the corresponding parametric solution set is presented numerically in Table 1 and graphically on Fig. 2 by dashed lines. Between all the degrees of freedom of the displacement, the 29th DOF showed maximal overestimation for all tolerance values. The computationally efficient iterative procedure gives sharp enclosures for small number of parameters and thin intervals for the uncertainties. However, the overestimation grows with increasing the width of the intervals (Fig. 2). As seen from the last two rows of Table 1, the hybrid interval approach gives very tight enclosure of the exact hull for the displacement solution set.

Table 1. Overestimation, produced by different computing methods. $U_i^{(1)}$ is the % by which the hybrid interval bounding overestimates the i-th component of the exact hull to the PSS of (6). $U_i^{(2)}$ is the % by which Rump's fixed point iteration overestimates the i-th component of the solution to (6) obtained by the hybrid interval bounding. See Fig. 2

	parameter tolerances in %											
	0.01	0.1	1	2	3	4	5	6	7	8	9	10
$U_{42}^{(2)}$	0.02	0.2	2.0	3.98	5.94	7.88	9.81	11.72	13.61	15.52	17.40	19.27
$U_{29}^{(2)}$	0.26	2.55	21.36	35.99	46.74	54.85	61.20	66.28	70.45	74.05	76.99	79.51
$U_{42}^{(1)}$	$5{*}10^{-7}$	$5{*}10^{-8}$	$5{*}10^{-9}$	$2{*}10^{-9}$	$2{*}10^{-9}$	$1{*}10^{-9}$	$1{*}10^{-9}$	$8{*}10^{-10}$	$7{*}10^{-10}$	$6{*}10^{-10}$	$5{*}10^{-10}$	$5{*}10^{-10}$
$U_{29}^{(1)}$	$1{*}10^{-5}$	$1{*}10^{-6}$	$1{*}10^{-7}$	$8{*}10^{-8}$	$5{*}10^{-8}$	$4{*}10^{-8}$	$3{*}10^{-8}$	$3{*}10^{-8}$	$2{*}10^{-8}$	$2{*}10^{-8}$	$2{*}10^{-8}$	$2{*}10^{-8}$

4.2 Bounding Stresses and Strains

We fix the level of uncertainty in the material properties at 5% and apply the hybrid interval approach, presented in Section 3.2, to the computation of the element strains and the element stresses.

For the elements around the node 21 (see Fig. 1), a straightforward interval evaluation of (8) and (9) was done by using interval estimations for the element displacements. Two interval estimations for the element displacements were used: interval enclosure, obtained by the parametric iteration method, and the enclosure of the exact interval hull by the hybrid approach. The results for the element strains, with corresponding numbers 24, 25 and 26 (see Fig. 1), are presented in Table 2. A huge overestimation of the strains range is demonstrated, which is due to the parameter dependencies between the components of U_e. The dependency, accumulated in the interval enclosure $[U_e]$ cannot be taken into account by the interval computation $B \cdot [U_e]$. It is remarkable, that the difference in the interval estimations of U (by Rump's method and the exact hull) is obscured by the inherited dependencies between the displacement components.

The naive range estimation of the element stresses showed similar results. Note that the expression (9) involves additional dependencies due to the parameters in the matrix $D(p_e)$. Accounting for the dependencies in $D(p_e)$ reduces the overestimation by 30–50%.

Table 2. Overestimation, produced by naive range estimation of the element strains $\varepsilon(24), \varepsilon(25), \varepsilon(26)$. Two interval estimations for the displacements are used: (R) Rump's enclosure, and (M) enclosure obtained by the hybrid approach. (M/R) presents the ratio in % between the two displacements estimations.

$\varepsilon(24)$	1	2	3
	component		
R	57.33	81.04	98.32
M	0.08	78.86	98.09
M/R	57.30	10.33	12.21

$\varepsilon(25)$	1	2	3
	component		
R	89.14	81.04	94.82
M	75.53	78.86	93.92
M/R	55.60	10.33	14.75

$\varepsilon(26)$	1	2	3
	component		
R	62.45	81.71	96.38
M	1.47	79.50	95.85
M/R	61.84	10.81	12.81

As a result of the interval finite element analysis, the stress and strain ranges are obtained within each element of the finite element model. The corresponding interval values for the strains of the six neighbour finite elements, defined by node 21 (see Fig. 1), are presented on Fig. 3.

Based on the range of values for the stresses and strains at the neighbour finite elements, defined by a node, we can smooth the nodal stresses and strains. Two methods are possible:

– Analogously to the crisp case, the nodal strain (stress) is defined as the interval mean value of the neighbouring element strain (stress) ranges.
– The nodal strain (stress) is defined as the interval intersection of the neighbouring element strain (stress) ranges.

The two approaches for smoothing the nodal strains are represented on Fig. 3. The advantage of the second method is that the resulting nodal strain (stress) range possesses no excess points. This means that each point from the nodal range belongs to the ranges of all neighbouring element strains (stresses). The latter is not true for the nodal range, obtained as an interval mean value.

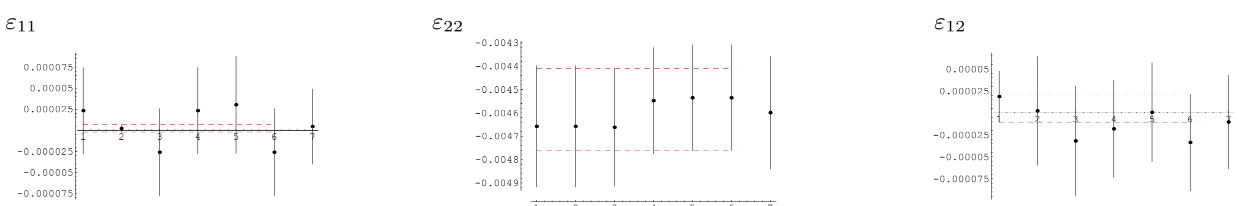


Fig. 3. Strains ranges for the elements that have a contact with the node 21: 24 (1), 25 (2), 26 (3), 33 (4), 34 (5), 35 (6). Dashed lines represent the bounds of the range intersection. The seventh interval represents the interval mean value of the six element ranges.

5 Conclusions

We presented modelling of composite material with inclusions where the elastic material properties of both matrix and inclusions were uncertain and varied within prescribed bounds. Several interval techniques were applied and compared in their efficiency to provide sharp enclosure for the corresponding solution sets. The sources for overestimation were demonstrated.

The hybrid interval approach, presented in Sec. 3, is based on numerical proof of monotonicity properties and on a fast iteration method for enclosing the parametric solution set. By using validated interval computations, this approach has the additional property that, owing to an automatic error control mechanism, every computed result is guaranteed to be correct. The hybrid approach provides very sharp bounds for the nodal displacements, as well as for the process variables – element strains and element stresses.

Based on the range of values for the stresses and strains at the neighbour finite elements, defined by a node, we proposed two smoothing methods for the nodal stresses and strains.

The interval finite element methods, presented in this paper, can be applied to all engineering problems whose behavior is governed by parametric interval linear systems and response quantities of type (7).

The *Mathematica* package [3] should be extended by suitable functions automating the computations of the response quantities. Further research toward improving the computational efficiency of the method would allow large industrial applications.

References

1. Chen, D.; Moler, C.: Symbolic Matlab Toolbox User's Guide. The Math Works Inc., 1994.
2. Muhanna R.; Mullen R.: Short Review for the Latest Development of Interval FEM. 2002. `http://www.gtrep.gatech.edu/~rmuhanna/IFEM-review.pdf`
3. Popova, E.: Parametric Interval Linear Solver. Numerical Algorithms, submitted.
4. Rao S. S.; Berke L.: Analisys of Uncerttain Structural Systems Using Interval Analysis. AIAA Journal, 35 (1997) 4, 727-735.
5. Rohn, J.: Interval Linear Equations with Dependent Coefficients. Reliable Computing mailing list, 2002. `http://www.ms.mff.cuni.cz/~rohn/letter/letter.ps`
6. Rump, S.: Verification Methods for Dense and Sparse Systems of Equations. In: Herzberger, J.(ed.): Topics in Validated Computations. Elsevier Science B. V. (1994) 63-135.
7. Schanz T.: Zur Modellierung des mechanischen Verhaltens von Reibungsmaterialien. Habilitationsschrift, Institut fuer Geotechnik, Stuttgart, 1998.
8. Washizu K.: Variational Methods in Elasticity and Plasticity. Pergamon Press, New York, 1982.
9. S. Wolfram, The Mathematica Book, 4th ed., Wolfram Media/Cambridge U. Press, 1999.
10. Zienkiewicz O. C.: The Finite Element Method in Engineering Science. McGraw-Hill, London, 1971.